

## PRINCIPAL CURVATURES OF ISOPARAMETRIC HYPERSURFACES IN $\mathbb{C}P^n$

LIANG XIAO

ABSTRACT. Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$ , and  $\overline{M}$  the inverse image of  $M$  under the Hopf map. By using the relationship between the eigenvalues of the shape operators of  $M$  and  $\overline{M}$ , we prove that  $M$  is homogeneous if and only if either  $g$  or  $l$  is constant, where  $g$  is the number of distinct principal curvatures of  $M$  and  $l$  is the number of non-horizontal eigenspaces of the shape operator on  $\overline{M}$ .

### INTRODUCTION

A compact hypersurface  $M$  in a compact symmetric space is called *isoparametric* if all parallel hypersurfaces  $M_t$  for  $t$  sufficiently close to zero have constant mean curvatures. Wang [15], Kimura [4] and Park [8] have studied isoparametric hypersurfaces in  $\mathbb{C}P^n$ . For instance, Wang [15] proved that a hypersurface  $M$  in  $\mathbb{C}P^n$  is isoparametric if and only if its inverse image  $\pi^{-1}(M)$  under the Hopf map is isoparametric in  $S^{2n+1}$ . This allows us to study the geometry of an isoparametric hypersurface  $M$  in  $\mathbb{C}P^n$  by studying the geometry of the isoparametric hypersurfaces  $\overline{M} = \pi^{-1}(M)$  in  $S^{2n+1}$ . Here  $\pi : S^{2n+1} \rightarrow \mathbb{C}P^n$  is the well known Hopf map. Isoparametric hypersurfaces in  $\mathbb{C}P^n$  have many interesting and remarkable phenomena, some of which are quite different from the case in spheres. For example, there are many isoparametric hypersurfaces in  $\mathbb{C}P^n$  whose principal curvatures are not constant [15]; an isoparametric hypersurface  $M$  in  $\mathbb{C}P^n$  has constant principal curvatures if and only if  $M$  is homogeneous [4]. In [8], Park shows that there are two noncongruent isoparametric hypersurfaces in  $\mathbb{C}P^3$  with congruent homogeneous inverse images under the Hopf map. In [16], the author found many isoparametric submanifolds (including hypersurfaces) in  $\mathbb{C}P^n$ . Among the examples, we find  $k$  mutually noncongruent isoparametric hypersurfaces in  $\mathbb{C}P^{2m+1}$  with congruent homogeneous inverse images under the Hopf map for any  $k \geq 3$ . Here  $m \geq \max(2k - 5, 2)$ . In particular, the author [16] obtained many isoparametric hypersurfaces which are not homogeneous but have homogeneous inverse images. In the 1990s, Terng and Thorbergsson [14] founded the deep theory of equifocal submanifolds in symmetric spaces. According to their theory, an equifocal hypersurface in  $\mathbb{C}P^n$  is exactly an isoparametric hypersurface.

Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$ , and  $\overline{M}$  the inverse image of  $M$  under the Hopf map. Let  $g$  denote the number of distinct principal curvatures of  $M$  and  $l$  the number of non-horizontal eigenspaces of the shape operator on  $\overline{M}$ .

---

Received by the editors December 20, 1998 and, in revised form, March 25, 1999.  
 2000 *Mathematics Subject Classification*. Primary 53C40.

In [8], Park tried to give a classification of isoparametric hypersurfaces in  $\mathbb{C}P^n$ . But unfortunately, the statement that both  $g$  and  $l$  are constant for any isoparametric hypersurface  $M$  in  $\mathbb{C}P^n$ , which Theorems A and B and Table 1 in [8] heavily depend on, is incorrect because Propositions 3.12 and 3.13 are not true. Some counterexamples will be showed in Propositions 2.3 and 2.4 in this paper. So one natural question to ask is, when does an isoparametric hypersurface  $M$  have constant  $g$  or constant  $l$ ? In this paper, we give a complete solution to this question. The main theorem in this paper is the following:

**Theorem.** *Let  $M$  be a complete isoparametric hypersurface in  $\mathbb{C}P^n$ , and  $\overline{M}$  the inverse image of  $M$  under the Hopf map. Then  $M$  is homogeneous if and only if either  $g$  or  $l$  is constant.*

The paper has two sections. In Section 1, we recall the basic definitions and the relation between the shape operators of a hypersurface  $M$  in  $\mathbb{C}P^n$  and its inverse image  $\overline{M}$  under the Hopf map. In Section 2, we prove our main theorem by showing that if  $M$  is an isoparametric hypersurface, but not homogeneous, in  $\mathbb{C}P^n$ , then neither  $g$  nor  $l$  is constant.

This work was done when the author was a visiting scholar at Northeastern University in Boston. He wishes to express his gratitude to Professor Chu-Lian Terng for her many helpful conversations and support. He is grateful to Professor Chia-Kuei Peng for his long term guidance and help.

## 1. PRELIMINARIES

In this section we recall the basic properties of the Hopf map  $\pi : S^{2n+1} \rightarrow \mathbb{C}P^n$  and relation between shape operators of the hypersurfaces  $M$  in  $\mathbb{C}P^n$  and  $\overline{M} = \pi^{-1}(M)$ . Then we recall the theory of isoparametric hypersurfaces in  $\mathbb{C}P^n$  and  $S^{2n+1}$ , which were first studied in [1], [5], [6], [8].

Let  $\mathbb{C}^{n+1}$  be an  $(n+1)$ -dimensional complex space, and  $\mathbb{C}P^n$  the complex projective space, which is given the Fubini-Study metric of constant holomorphic sectional curvature 4. Denote by  $S^{2n+1}$  the unit sphere in  $\mathbb{C}^{n+1}$  defined by  $z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_{n+1}\bar{z}_{n+1} = 1$ , and by  $S^1$  the multiplicative group of complex numbers of absolute value 1. As is well known,  $S^{2n+1}$  is a principal fibre bundle over  $\mathbb{C}P^n$  with group  $S^1$ ; the projection  $\pi : S^{2n+1} \rightarrow \mathbb{C}P^n$  is the Hopf map and Riemannian submersion; each fibre of  $\pi$  is a geodesic, and for any  $x$  in  $S^{2n+1}$ ,  $\overline{J}x$  is a tangent vector of the fibre through  $x$ . Here  $\overline{J}$  is the natural complex structure of  $\mathbb{C}^{n+1}$ .

Denote the Levi-Civita connections of  $S^{2n+1}$  and  $\mathbb{C}P^n$  by  $\overline{\nabla}$  and  $\nabla$ , respectively. Then

$$\mathcal{H}(\overline{\nabla}_X \overline{Y}) = \overline{\nabla}_X \overline{Y}$$

for any vector fields  $X$  and  $Y$  on  $\mathbb{C}P^n$ . Here  $\overline{X}$ ,  $\overline{Y}$  and  $\overline{\nabla}_X \overline{Y}$  mean their horizontal lifts, and  $\mathcal{H}$  means the horizontal projection.

Let  $M$  be a real hypersurface of  $\mathbb{C}P^n$ ,  $\xi$  a unit normal vector field on  $M$ . Then the horizontal lift  $\overline{\xi}$  of  $\xi$  is the unit normal vector field on  $\overline{M} = \pi^{-1}(M)$ , and the relationship between the two shape operators  $A_{\overline{\xi}}$  and  $A_{\xi}$  is given by

$$(1) \quad A_{\overline{\xi}} \overline{Y} = \overline{A_{\xi} Y} + \langle A_{\overline{\xi}} \overline{Y}, \overline{J}X \rangle \overline{J}X,$$

$$(2) \quad A_{\overline{\xi}} \overline{J}X = \overline{J} \xi.$$

Here  $X$  is the position vector field on  $\overline{M}$ ,  $Y$  a vector field on  $M$ , and  $\overline{J}$  the complex structure of  $\mathbb{C}^{n+1}$ .

Recall that a connected compact real hypersurface  $M$  in  $\mathbb{C}P^n$  is isoparametric if and only if  $\overline{M} = \pi^{-1}(M)$  is isoparametric in  $S^{2n+1}$  [15]. Isoparametric hypersurfaces in  $S^n$  are well studied in [1], [5], [6] by Abresch and Münzner. Some results of [1], [5], [6], reformulated in our terminology, are given in the following theorem:

**Theorem 1.1** ([1], [5], [6]). *Let  $M$  be an isoparametric hypersurface in  $S^n$  with  $g$  distinct constant principal curvatures  $\lambda_1 > \dots > \lambda_g$  along the unit normal vector  $\xi$  with multiplicities  $m_1, \dots, m_g$ , and  $E_j$  the curvature distribution defined by  $\lambda_j$ , i.e.,  $E_j(x)$  is equal to the eigenspace of  $A_{\xi(x)}$  with respect to the eigenvalue  $\lambda_j$ . Then:*

- (1) *The number  $g = 1, 2, 3, 4$  and 6.*
- (2) *There exists  $0 < \theta < \frac{\pi}{g}$  such that  $\lambda_j = \cot(\theta + \frac{(j-1)\pi}{g})$  with  $j$  in  $\{1, \dots, g\}$ .*
- (3) *Each distribution  $E_j$  is integrable, and its integral manifold is a sphere with dimension  $m_j$ .*
- (4) *If  $g = 3$ , then  $m = m_1 = m_2 = m_3 = 1, 2, 4$  or 8.*
- (5) *If  $g = 4$ , then  $m_1 = m_3, m_2 = m_4$  and one of  $m_1$  and  $m_2$  is 1 or even.*
- (6) *If  $g = 6$ , then  $m = m_1 = \dots = m_6 = 1$  or 2.*

Let  $M$  be an isoparametric hypersurface in  $S^{2n+1} \subset \mathbb{C}^{n+1}$ ,  $\xi$  a unit normal vector field on  $M$ . Then for any  $x \in M$ , an eigenspace  $E_i(x)$  of  $A_{\xi(x)}$  is called horizontal if all vectors in  $E_i(x)$  are horizontal. Otherwise, it is called nonhorizontal.

In [4], Kimura proved the following:

**Theorem 1.2** ([4]). *Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$ . Then  $M$  is homogenous if and only if it has constant principal curvatures.*

Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$ , and  $\overline{M}$  its inverse image under the Hopf map  $\pi$ . Denote the number of nonhorizontal eigenspaces  $E_i(x)$  of  $A_{\xi(x)}$  by  $l(x)$  for any  $x \in \overline{M}$ . Since  $\overline{M}$  is  $S^1$ -invariant,  $l$  can be viewed as a function of  $M$ . Since the errors of Proposition 3.12 to 3.13 in [8] do not influence the validity of all results and proofs from Propositions 3.1–3.11 in [8], the following theorems are known:

**Theorem 1.3** ([8]). *Let  $\overline{M}$  be an inverse image of an isoparametric hypersurface  $M$  in  $\mathbb{C}P^n$ . Then  $\overline{M}$  is an isoparametric hypersurface in  $S^{2n+1}$  and:*

- (1) *The number  $\overline{g}$  of distinct principal curvatures of  $\overline{M}$  is 2, 4 or 6.*
- (2) *If  $\overline{g} = 2$ , then both  $m_1$  and  $m_2$  are odd numbers.*
- (3) *If  $\overline{g} = 4$ , then either  $\min(m_1, m_2) = 1$  or  $m_1 m_2$  is even.*
- (4) *If  $\overline{g} = 6$ , then  $m = 1$  and  $M$  is not homogeneous.*

**Theorem 1.4** ([8]). *Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$ . Then  $l \equiv 2$  if and only if  $M$  has constant principal curvatures.*

As a consequence of Theorem 1.2 and Theorem 1.4, we have:

**Theorem 1.5.** *Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$ . Then  $l \equiv 2$  if and only if  $M$  is homogeneous.*

## 2. PROOF OF THE MAIN THEOREM

In this section we discuss the number  $g$  of distinct principal curvatures of  $M$  in  $\mathbb{C}P^n$  and prove our main theorem, whose proof is based on the following propositions.

Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$ ,  $\overline{M}$  the inverse image of  $M$ , and  $\overline{\lambda}_1, \overline{\lambda}_2, \dots, \overline{\lambda}_{\overline{g}}$  all distinct principal curvatures of  $\overline{M}$ . Without loss of generality, we may assume that  $\overline{\lambda}_j = \cot(\theta + \frac{(j-1)\pi}{\overline{g}})$ ,  $0 < \theta < \frac{\pi}{\overline{g}}$ ,  $j = 1, \dots, \overline{g}$ . By Theorem 1.3, we know that  $\overline{g}$  is 2, 4 or 6.

**Proposition 2.1.** *Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$  ( $n > 1$ ). Assume that  $\overline{g} = 2$ . Then:*

- (1) *If  $m_1 = 1$  (or  $m_2 = 1$ ), then  $M$  has 2 constant principal curvatures  $\overline{\lambda}_1 + \overline{\lambda}_2$  and  $\overline{\lambda}_2$  with multiplicities 1 and  $m_2 - 1$  (or  $\overline{\lambda}_1$  and  $\overline{\lambda}_1 + \overline{\lambda}_2$  with multiplicities  $m_1 - 1$  and 1), respectively.*
- (2) *If  $m_1, m_2 > 1$ , then  $M$  has 3 constant principal curvatures  $\overline{\lambda}_1 + \overline{\lambda}_2$ ,  $\overline{\lambda}_1$  and  $\overline{\lambda}_2$  with multiplicities 1,  $m_1 - 1$  and  $m_2 - 1$ , respectively.*

*Proof.* Let  $E_1$  and  $E_2$  be the curvature distributions defined by  $\overline{\lambda}_1$  and  $\overline{\lambda}_2$ , respectively. Choose an orthonormal frame  $e_1, \dots, e_m, e_{m+1}, \dots, e_{2n-1}$  on  $T\overline{M}$  such that  $e_1, \dots, e_m \in E_1$ ,  $e_{m+1}, \dots, e_{2n-1} \in E_2$  and  $\overline{J}X = a_1 e_1 + a_2 e_{m+1}$  for some nonnegative functions  $a_1$  and  $a_2$  of  $\overline{M}$ . Here  $X$  is a position vector field. Then from the definition of  $\overline{J}$ , we have

$$(3) \quad a_1^2 + a_2^2 = 1.$$

Since  $e_1 \in E_1$  and  $e_{m+1} \in E_2$ , we know that

$$(4) \quad A_{\overline{\xi}}(\overline{J}X) = \overline{\lambda}_1 a_1 e_1 + \overline{\lambda}_2 a_2 e_{m+1}.$$

It follows from (2) that

$$A_{\overline{\xi}}(\overline{J}X) = \overline{J} \overline{\xi}.$$

Hence

$$(5) \quad \overline{\lambda}_1 a_1^2 + \overline{\lambda}_2 a_2^2 = 0,$$

$$(6) \quad \overline{\lambda}_1^2 a_1^2 + \overline{\lambda}_2^2 a_2^2 = 1.$$

Combining (3), (5) and (6), we have

$$(7) \quad \overline{\lambda}_1 \overline{\lambda}_2 = -1,$$

$$(8) \quad a_1^2 = \frac{1}{1 + \overline{\lambda}_1^2},$$

$$(9) \quad a_2^2 = \frac{1}{1 + \overline{\lambda}_2^2}.$$

From (1), we know that  $\pi_*(-a_2 e_1 + a_1 e_{m+1}), \pi_* e_2, \dots, \pi_* e_m, \pi_* e_{m+2}, \dots, \pi_* e_{2n}$  form an orthogonal frame of  $M$  and each of them is a principal direction, and the corresponding principal curvatures are  $\overline{\lambda}_1 + \overline{\lambda}_2$ ,  $\overline{\lambda}_1$  and  $\overline{\lambda}_2$  with multiplicities 1,  $m_1 - 1$  and  $m_2 - 1$ , respectively.  $\square$

---

<sup>1</sup>The frame  $e_1, \dots, e_{2n}$  need not to be continuous.

**Proposition 2.2.** *Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$ , and  $\overline{M}$  the inverse image of  $M$ . Assume that  $\overline{g} = 4$  and  $m_1, m_2 > 1$ . Then:*

- (1) *There is an  $x_0 \in M$  such that  $l(x_0) = 2$ .*
- (2)  *$M$  is homogeneous if and only if  $M$  has 5 constant principal curvatures.*
- (3) *If  $M$  is not homogeneous, then  $M$  has 7 principal curvatures at general points and 5 principal curvatures at special points.*

*Proof.* Let  $E_1, E_2, E_3$  and  $E_4$  be the curvature distributions defined by  $\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3$  and  $\overline{\lambda}_4$ , respectively. Choose an orthonormal frame  $e_1, \dots, e_{2n}$  on  $T\overline{M}$  such that

$$\begin{aligned} e_1, \dots, e_{m_1} &\in E_1, & e_{m_1+1}, \dots, e_{m_1+m_2} &\in E_2, \\ e_{m_1+m_2+1}, \dots, e_{2m_1+m_2} &\in E_3, & e_{2m_1+m_2+1}, \dots, e_{2n} &\in E_4, \end{aligned}$$

and

$$\overline{J}X = a_1 e_1 + a_2 e_{m_1+1} + a_3 e_{m_1+m_2+1} + a_4 e_{2m_1+m_2+1}.$$

Note that  $\langle \overline{J}X, \overline{J}X \rangle = 1$ ,  $\langle \overline{J}X, X \rangle = 0$ , and

$$\overline{J}\xi = A_{\xi}(\overline{J}X) = \overline{\lambda}_1 a_1 e_1 + \overline{\lambda}_2 a_2 e_{m_1+1} + \overline{\lambda}_3 a_3 e_{m_1+m_2+1} + \overline{\lambda}_4 a_4 e_{2m_1+m_2+1}.$$

The functions  $a_i$  satisfy the following equations:

$$(10) \quad a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1,$$

$$(11) \quad \overline{\lambda}_1 a_1^2 + \overline{\lambda}_2 a_2^2 + \overline{\lambda}_3 a_3^2 + \overline{\lambda}_4 a_4^2 = 0,$$

$$(12) \quad \overline{\lambda}_1^2 a_1^2 + \overline{\lambda}_2^2 a_2^2 + \overline{\lambda}_3^2 a_3^2 + \overline{\lambda}_4^2 a_4^2 = 1.$$

Note that  $\overline{\lambda}_j = \cot(\theta + \frac{(j-1)\pi}{4})$  for all  $1 \leq j \leq 4$ . The equations (10), (11) and (12) imply that

$$(13) \quad a_1^2 = \frac{\cos^2 \phi}{1 + \overline{\lambda}_1^2}, \quad a_2^2 = \frac{\sin^2 \phi}{1 + \overline{\lambda}_2^2}, \quad a_3^2 = \frac{\cos^2 \phi}{1 + \overline{\lambda}_3^2}, \quad a_4^2 = \frac{\sin^2 \phi}{1 + \overline{\lambda}_4^2}$$

for some function  $\phi$  of  $\overline{M}$ . Hence  $l(x) = 2$  or  $4$  for any  $x$  in  $\overline{M}$ .

Now we prove that  $g = l + 3$ . If  $l(x) = 2$  and  $a_1 = a_3 = 0$ , then the principal curvatures of  $M$  at  $x$  are  $\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \overline{\lambda}_4$  and  $\overline{\lambda}_2 + \overline{\lambda}_4$ , which implies  $g(x) = 5$ . If  $l(x) = 2$  and  $a_2 = a_4 = 0$ , then the principal curvatures of  $M$  at  $x$  are  $\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \overline{\lambda}_4$  and  $\overline{\lambda}_1 + \overline{\lambda}_3$ , which also implies  $g(x) = 5$ . From (1) and (2), we know that if  $l(x) = 4$ , then the principal curvatures of  $M$  at  $x$  are  $\lambda_1, \lambda_2, \lambda_3, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3$ , and  $\overline{\lambda}_4$ , where  $\lambda_1, \lambda_2, \lambda_3$  are exactly the roots of the following equation on  $\lambda$ :

$$\sum_{i=1}^4 \frac{a_i^2}{\lambda - \overline{\lambda}_i} = 0.$$

Hence  $g(x) = 7$ .

Now we prove that there is an  $x_0$  in  $\overline{M}$  such that  $l(x_0) = 2$ . First, we note that  $\{a_1 e_1, a_2 e_{m_1+1}, a_3 e_{m_1+m_2+1}, a_4 e_{2m_1+m_2+1}\}$  are orthogonal projections of  $\overline{J}X$  in  $E_1, E_2, E_3$  and  $E_4$ , respectively, and that the integral manifolds of  $E_1, E_2, E_3$  and  $E_4$  are spheres with dimension  $m_1, m_2, m_1$  and  $m_2$ , respectively. By Theorem 1.1, we know that one of  $m_1$  and  $m_2$  must be even. Since any vector field on an even dimensional sphere must have zero points, we know that there is an  $x_0$  in  $M$  such that  $l(x_0) = 2$ . Therefore if  $M$  is homogeneous, then  $M$  has 5 constant principal

curvatures; and if  $M$  is not homogeneous, then  $M$  has 7 principal curvatures at general points and 5 principal curvatures at special points.  $\square$

**Proposition 2.3.** *Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$ , and  $\overline{M}$  the inverse image of  $M$ . Assume that  $\overline{g} = 4$  and  $m_1$  or  $m_2 = 1$ . Then:*

- (1)  *$M$  is homogeneous if and only if  $M$  has 3 constant principal curvatures.*
- (2) *If  $M$  is not homogeneous, then  $M$  has 5 principal curvatures at general points and 3 principal curvatures at special points.*

*Proof.* In [10] Takagi proved that  $\overline{M}$  corresponds to the isotropy representation of the symmetric space  $G/K = SO(n+3)/S(O(2) \times O(n+1))$  in which  $n = m_1 + m_2$ . The correspondence is given as follows [17].

The isotropy group  $K$  consists of all the following matrices:

$$(14) \quad K = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where  $A_1 \in O(2)$ ,  $A_2 \in O(n+1)$  and  $\det(A_1 A_2) = 1$ .

Let  $\mathfrak{o}(n+3) = \mathfrak{k} + \mathfrak{p}$  be the Cartan decomposition, where  $\mathfrak{k}$  consists of all the following skew-symmetric matrices:

$$(15) \quad B = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$$

(here  $X_1 \in \mathfrak{o}(2)$  and  $X_2 \in \mathfrak{o}(n+1)$ );  $\mathfrak{p}$  consists of all the following skew-symmetric matrices:

$$(16) \quad P = \begin{pmatrix} 0 & X \\ -X' & 0 \end{pmatrix},$$

where  $X$  is a  $2 \times (n+1)$  matrix.

The inner product on  $P$  is

$$(17) \quad \langle P_1, P_2 \rangle = -\frac{1}{2} \operatorname{tr} P_1 P_2 \quad (\text{where } P_1, P_2 \in P),$$

which turns out to be the standard Euclidean metric. Let  $S^{2n+1}$  be the unit sphere of  $\mathfrak{p}$ , and  $\overline{M}$  the orbit  $\operatorname{Ad} K \cdot P_0$  in  $S^{2n+1}$ , where

$$P_0 = \begin{pmatrix} 0 & X_0 \\ -X_0' & 0 \end{pmatrix},$$

and

$$(18) \quad X_0 = \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \end{pmatrix}.$$

Here

$$(x_1, x_2) = \begin{cases} \left( \frac{\sqrt{2}}{2} \frac{1 + \overline{\lambda}_1}{(1 + \overline{\lambda}_1^2)^{\frac{1}{2}}}, \frac{\sqrt{2}}{2} \frac{\overline{\lambda}_1 - 1}{(1 + \overline{\lambda}_1^2)^{\frac{1}{2}}} \right), & \text{if } m_1 = 1, \\ \left( \frac{1}{(1 + \overline{\lambda}_1^2)^{\frac{1}{2}}}, \frac{\overline{\lambda}_1}{(1 + \overline{\lambda}_1^2)^{\frac{1}{2}}} \right), & \text{if } m_2 = 1. \end{cases}$$

The normal vector of  $\overline{M}$  in  $S^{2n+1}$  at  $P_0$  is

$$\begin{cases} \left(-\frac{\sqrt{2}}{2} \frac{\overline{\lambda}_1 - 1}{(1 + \overline{\lambda}_1^2)^{\frac{1}{2}}}, \frac{\sqrt{2}}{2} \frac{\overline{\lambda}_1 + 1}{(1 + \overline{\lambda}_1^2)^{\frac{1}{2}}}\right), & \text{if } m_1 = 1, \\ \left(-\frac{\overline{\lambda}_1}{(1 + \overline{\lambda}_1^2)^{\frac{1}{2}}}, \frac{1}{(1 + \overline{\lambda}_1^2)^{\frac{1}{2}}}\right), & \text{if } m_2 = 1. \end{cases}$$

At the point  $kP_0k^{-1}$ , the curvature distributions  $E_i$  ( $i = 1, \dots, 4$ ) of  $\overline{M}$  consist of the following matrices:

$$k \begin{pmatrix} 0 & Y_i \\ -Y_i' & 0 \end{pmatrix} k^{-1},$$

where

$$Y_1 = \begin{pmatrix} 0 & x & 0 & \cdots & 0 \\ x & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & x_1 & \cdots & x_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} 0 & x & 0 & \cdots & 0 \\ -x & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Y_4 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_1 & \cdots & x_{n-1} \end{pmatrix}$$

for  $m_1 = 1$ , and

$$Y_1 = \begin{pmatrix} 0 & 0 & x_1 & \cdots & x_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & x & 0 & \cdots & 0 \\ -x & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_1 & \cdots & x_{n-1} \end{pmatrix}, \quad Y_4 = \begin{pmatrix} 0 & x & 0 & \cdots & 0 \\ x & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for  $m_2 = 1$ . All  $Y_1, Y_2, Y_3, Y_4$  here are real  $2 \times (n+1)$  matrices.

Now we determine the complex structure  $\overline{\mathcal{J}}$  of  $\mathfrak{p}$ . It is proved in [16] that the structural group  $S^1 \subset \text{Ad } K|_{\mathfrak{p}}$  and the complex structure  $\overline{\mathcal{J}} = \text{ad } A$  for some  $A \in \mathfrak{k}$  and  $A$  satisfies the following equations:

$$(19) \quad \langle \text{ad } A \cdot P_1, \text{ad } A \cdot P_2 \rangle = \langle P_1, P_2 \rangle,$$

$$(20) \quad (\text{ad } A)^2 \cdot P = -P$$

for any  $P, P_1$  and  $P_2$  in  $\mathfrak{p}$ . Hence

$$A = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & \cdots & 0 \\ & & \vdots & & \vdots \\ & & 0 & \cdots & 0 \end{pmatrix},$$

or  $n$  is odd and

$$A = k \begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & \ddots & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix} k^{-1}$$

for some  $k$  in  $K$  [16].

If

$$A = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & \cdots & 0 \\ & & \vdots & & \vdots \\ & & 0 & \cdots & 0 \end{pmatrix},$$

then  $M$  is homogeneous with  $l = 2$  and  $g = 3$ ; if  $n$  is odd and

$$A = k \begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix} k^{-1}$$

for some  $k$  in  $K$ , then  $M$  is not homogeneous. Choose a point

$$x_0 = k \begin{pmatrix} 0 & X_0 \\ -X'_0 & 0 \end{pmatrix} k^{-1},$$

where  $X_0$  is as in (18). Then  $l(x_0) = 2$  and  $g(\pi(x_0)) = 3$ . It is easy to see that for general points  $x$  of  $M$  in  $\mathbb{C}P^n$ ,  $l(x) = 4$  and  $g(x) = 5$ .  $\square$

**Proposition 2.4.** *Let  $M$  be an isoparametric hypersurface in  $\mathbb{C}P^n$ , and  $\overline{M}$  the inverse image of  $M$ . Assume that  $\overline{g} = 6, m = 1$ . Then:*

- (1)  $l$  is not constant.
- (2)  $M$  has 5 principal curvatures at general points and 4 principal curvatures at special points.

To prove Proposition 2.4, we need the following:

**Lemma 2.5.** *Assume that  $\overline{\lambda}_j = \cot(\theta + \frac{(j-1)\pi}{6})$ ,  $j = 1, \dots, 6$ ,  $0 < \theta < \frac{\pi}{6}$ . Then*

- (1)  $\frac{\sqrt{3}}{3} < \overline{\lambda}_2 < \sqrt{3}$ ,  $\overline{\lambda}_3 < \overline{\lambda}_1 + \overline{\lambda}_4$ ,  $\overline{\lambda}_4 > \overline{\lambda}_3 + \overline{\lambda}_6$ ;
- (2)  $\overline{\lambda}_3 > \overline{\lambda}_2 + \overline{\lambda}_5$  if  $\frac{\sqrt{3}}{3} < \overline{\lambda}_2 < \sqrt[6]{3}$ ; and
- (3)  $\overline{\lambda}_4 < \overline{\lambda}_2 + \overline{\lambda}_5$  if  $\frac{1}{\sqrt[6]{3}} < \overline{\lambda}_2 < \sqrt{3}$ .

*Proof.* Since

$$\begin{aligned} \sqrt{3} < \overline{\lambda}_1 < +\infty, \quad \frac{\sqrt{3}}{3} < \overline{\lambda}_2 < \sqrt{3}, \quad 0 < \overline{\lambda}_3 < \frac{\sqrt{3}}{3}, \\ -\frac{\sqrt{3}}{3} < \overline{\lambda}_4 < 0, \quad -\sqrt{3} < \overline{\lambda}_5 < -\frac{\sqrt{3}}{3}, \quad -\infty < \overline{\lambda}_6 < -\sqrt{3}, \end{aligned}$$

we have

$$\overline{\lambda}_3 < \frac{\sqrt{3}}{3} < \sqrt{3} - \frac{\sqrt{3}}{3} < \overline{\lambda}_1 + \overline{\lambda}_4,$$

and

$$\overline{\lambda}_4 > -\frac{\sqrt{3}}{3} > \frac{\sqrt{3}}{3} - \sqrt{3} > \overline{\lambda}_3 + \overline{\lambda}_6.$$



Hence (1) holds. Note that

$$\bar{\lambda}_3 = \frac{\sqrt{3}\bar{\lambda}_2 - 1}{\bar{\lambda}_2 + \sqrt{3}}, \quad \bar{\lambda}_4 = \frac{\bar{\lambda}_2 - \sqrt{3}}{\sqrt{3}\bar{\lambda}_2 + 1}, \quad \bar{\lambda}_5 = -\frac{1}{\bar{\lambda}_2}.$$

The inequalities  $\bar{\lambda}_3 > \bar{\lambda}_2 + \bar{\lambda}_5$  and  $\bar{\lambda}_4 < \bar{\lambda}_2 + \bar{\lambda}_5$  are equivalent to  $\bar{\lambda}_2^3 < \sqrt{3}$  and  $\bar{\lambda}_2^3 > \frac{1}{\sqrt{3}}$ , respectively. So (2) and (3) hold.  $\square$

Now we are ready for the

*Proof of Proposition 2.4.* In [3], Dorfmeister and Neher proved that  $\overline{M}$  corresponds to the isotropy representation of symmetric space  $G/K = G_2/SU(2) \times SU(2)$ . The correspondence is given as follows [9].

Let  $\mathfrak{g}_2 = \mathfrak{k} + \mathfrak{p}$  be the Cartan decomposition, where  $\mathfrak{k}$  consists of all the following matrices:

$$(21) \quad B = \begin{pmatrix} E + F & 0 \\ 0 & G \end{pmatrix},$$

in which

$$(22) \quad E = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & -u_3 & u_2 \\ -u_2 & u_3 & 0 & -u_1 \\ -u_3 & -u_2 & u_1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -v_1 & -v_2 & -v_3 \\ v_1 & 0 & -v_3 & v_2 \\ v_2 & v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} 0 & -2v_3 & 2v_2 \\ 2v_3 & 0 & -2v_1 \\ -2v_2 & 2v_1 & 0 \end{pmatrix}$$

(here  $u_i$  and  $v_i$  are real numbers);  $\mathfrak{p}$  consists of all the following symmetric matrices:

$$(23) \quad P = \begin{pmatrix} 0 & 0 & \sqrt{2}Y \\ 0 & 0 & T - S \\ \sqrt{2}Y' & T + S & 0 \end{pmatrix}.$$

in which  $Y = \frac{1}{\sqrt{3}}(y_1, y_2, y_3)$ , and

$$(24) \quad T = \begin{pmatrix} x_1 & \frac{1}{\sqrt{2}}y_4 & \frac{1}{\sqrt{2}}y_5 \\ \frac{1}{\sqrt{2}}y_4 & x_2 & \frac{1}{\sqrt{2}}y_6 \\ \frac{1}{\sqrt{2}}y_5 & \frac{1}{\sqrt{2}}y_6 & x_3 \end{pmatrix}, \quad S = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}$$

(here  $x_i$  ( $i = 1, 2, 3$ ),  $y_j$  ( $j = 1, \dots, 6$ ) are real numbers and  $x_1 + x_2 + x_3 = 0$ ). The inner product of  $\mathfrak{p}$  is

$$(25) \quad \langle P_1, P_2 \rangle = -\frac{1}{2} \operatorname{tr} P_1 P_2 \quad (\text{where } P_1, P_2 \in \mathfrak{p}),$$

which turns out to be the standard Euclidean metric. Denote by  $S^7$  the unit sphere of  $\mathfrak{p}$ . Then  $\overline{M}$  is the orbit  $\operatorname{Ad} K \cdot P_0$  where

$$(26) \quad P_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & X_0 \\ 0 & -X_0 & 0 \end{pmatrix},$$

and

$$(27) \quad X_0 = \begin{pmatrix} \sqrt{\frac{2}{3(1+\lambda_1^2)}} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3(1+\lambda_5^2)}} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3(1+\lambda_3^2)}} \end{pmatrix}.$$

It is easy to see that the normal vector of  $\overline{M}$  in  $S^7$  at  $P_0$  is

$$(28) \quad \begin{pmatrix} -\sqrt{\frac{2}{3(1+\lambda_4^2)}} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3(1+\lambda_2^2)}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3(1+\lambda_6^2)}} \end{pmatrix},$$

and the principal directions of  $\overline{M}$  at  $P_0$  are

$$e_i = \begin{pmatrix} 0 & X_i \\ X'_i & 0 \end{pmatrix},$$

where

$$X_1 = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & 0 \\ -\frac{1}{\sqrt{6}} & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we calculate the complex structure  $\overline{J}$  of  $\mathbf{p}$ . It is proved in [16] that there is an  $A \in \mathbf{k}$  such that the complex structure  $\overline{J} = \text{ad } A$ , and  $A$  satisfies (19) and (20). Since  $A \in \mathbf{k}$ , we can find a  $k$  in  $K$  such that

$$(29) \quad kAk^{-1} = \begin{pmatrix} 0 & x+y & & & & \\ -x-y & 0 & & & & \\ & & 0 & -x+y & & \\ & & x-y & 0 & & \\ & & & & 0 & \\ & & & & & 0 & 2y \\ & & & & & -2y & 0 \end{pmatrix}$$

for some real  $x$  and  $y$ . Note that  $kAk^{-1}$  satisfies (19) and (20) for any  $k \in K$ . Substituting (29) into (19) and (20), we have  $x^2 = 1$  and  $y = 0$ . Hence

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where  $A_0$  is a  $4 \times 4$  skew orthonormal matrix. However, we notice that  $\overline{M}$  and  $\mathbf{p}$  are  $\text{Ad } k$ -invariant for any  $k \in \mathbf{k}$ . Without loss of generality, we may assume that the complex structure  $\overline{J} = \text{Ad } A$ , where

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \in o(7),$$

and

$$(30) \quad A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For any  $X$  in  $\overline{M}$ , write

$$\overline{J}X = \sum_{i=1}^6 a_i e_i.$$

Now we determine the possible values of  $a_i$ . Consider all the points  $kP_0k^{-1}$  at which

$$(31) \quad k = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} \in K,$$

and

$$(32) \quad R = \begin{pmatrix} b_1 & 0 & b_2 & b_3 \\ 0 & b_1 & -b_3 & b_2 \\ -b_2 & b_3 & b_1 & 0 \\ -b_3 & -b_2 & 0 & b_1 \end{pmatrix}$$

(here  $b_1, b_2$  and  $b_3$  are real and  $b_1^2 + b_2^2 + b_3^2 = 1$ ). At the  $kP_0k^{-1}$ ,

$$\begin{aligned} & (a_1, a_2, a_3, a_4, a_5, a_6) \\ &= \left( \frac{\sqrt{6}}{2} x_1 (2b_1^2 - 1), \sqrt{2} (x_1 - x_3) b_1 b_3, \sqrt{6} x_3 b_1 b_2, \right. \\ & \quad \left. \frac{1}{\sqrt{2}} (x_2 - x_3) (2b_1^2 - 1), -\sqrt{6} x_2 b_1 b_3, \sqrt{2} (x_1 - x_2) b_1 b_2 \right), \end{aligned}$$

where

$$x_1 = \sqrt{\frac{2}{3(1 + \overline{\lambda}_1)}}, \quad x_2 = \sqrt{\frac{2}{3(1 + \overline{\lambda}_5)}}, \quad x_3 = -\sqrt{\frac{2}{3(1 + \overline{\lambda}_3)}}.$$

Note that  $\overline{\lambda}_j = \cot(\theta + \frac{(j-1)\pi}{6})$  ( $j = 1, \dots, 6$ ) for some  $\theta$  ( $0 < \theta < \frac{\pi}{6}$ ). Hence

$$\begin{aligned} a_1^2 &= \frac{c_1^2}{1 + \overline{\lambda}_1}, & a_2^2 &= \frac{c_2^2}{1 + \overline{\lambda}_2}, & a_3^2 &= \frac{c_3^2}{1 + \overline{\lambda}_3}, \\ a_4^2 &= \frac{c_1^2}{1 + \overline{\lambda}_4}, & a_5^2 &= \frac{c_2^2}{1 + \overline{\lambda}_5}, & a_6^2 &= \frac{c_3^2}{1 + \overline{\lambda}_6} \end{aligned}$$

for some real  $c_1, c_2$  and  $c_3$ . Conversely, for any given real number  $c_1, c_2$  and  $c_3$  satisfying  $c_1^2 + c_2^2 + c_3^2 = 1$ , we can find an  $x$  in  $\overline{M}$  such that

$$(33) \quad \begin{aligned} a_1^2(x) &= \frac{c_1^2}{1 + \overline{\lambda}_1}, & a_2^2(x) &= \frac{c_2^2}{1 + \overline{\lambda}_2}, & a_3^2(x) &= \frac{c_3^2}{1 + \overline{\lambda}_3}, \\ a_4^2(x) &= \frac{c_1^2}{1 + \overline{\lambda}_4}, & a_5^2(x) &= \frac{c_2^2}{1 + \overline{\lambda}_5}, & a_6^2(x) &= \frac{c_3^2}{1 + \overline{\lambda}_6}. \end{aligned}$$

Therefore  $l(x) = 2, 4$  or  $6$ , and part (1) of Proposition 2.4 holds.

If  $l(x) = 6$ , then it follows from (1) and (2) that  $M$  has five principal curvatures, and the five principal curvatures are exactly the roots of the following equation on  $\lambda$ :

$$(34) \quad \sum_{i=1}^6 \frac{a_i^2}{\lambda - \bar{\lambda}_i} = 0.$$

Since  $\sqrt[6]{3} < \bar{\lambda}_2 < \sqrt{3}$ , we know that either  $\frac{1}{\sqrt[6]{3}} < \bar{\lambda}_2 < \sqrt{3}$  or  $\frac{\sqrt{3}}{3} < \bar{\lambda}_2 < \sqrt[6]{3}$  holds.

Without loss of generality, we may assume that  $\frac{1}{\sqrt[6]{3}} < \bar{\lambda}_2 < \sqrt[6]{3}$ . Now, we prove that there is an  $x_0$  in  $\bar{M}$  such that  $\bar{\lambda}_3$  is a principal curvature with multiplicity 2 at  $x_0$ .

Consider all the points  $x$  in  $\bar{M}$  such that

$$\begin{aligned} a_1^2(x) &= \frac{\sin^2 \phi}{1 + \bar{\lambda}_1^2}, & a_2^2(x) &= \frac{\cos^2 \phi}{1 + \bar{\lambda}_2^2}, & a_3(x) &= 0, \\ a_4^2(x) &= \frac{\sin^2 \phi}{1 + \bar{\lambda}_4^2}, & a_5^2(x) &= \frac{\cos^2 \phi}{1 + \bar{\lambda}_5^2}, & a_6(x) &= 0 \end{aligned}$$

for some  $\phi$  satisfying  $\sin \phi \neq 0$ , and  $\cos \phi \neq 0$ . Then the principal curvatures of  $M$  at  $\pi(x)$  are  $\lambda_1, \lambda_2, \lambda_3, \bar{\lambda}_3$  and  $\bar{\lambda}_6$ , where  $\lambda_1, \lambda_2$  and  $\lambda_3$  are exactly the roots of the following equation on  $\lambda$ :

$$(35) \quad \frac{a_1^2}{\lambda - \bar{\lambda}_1} + \frac{a_2^2}{\lambda - \bar{\lambda}_2} + \frac{a_4^2}{\lambda - \bar{\lambda}_4} + \frac{a_5^2}{\lambda - \bar{\lambda}_5} = 0.$$

So  $\bar{\lambda}_3$  is a root of (35) if  $\phi$  is a solution of the following equation:

$$(36) \quad \begin{aligned} & \left( \frac{1}{(1 + \bar{\lambda}_1^2)(\bar{\lambda}_3 - \bar{\lambda}_1)} + \frac{1}{(1 + \bar{\lambda}_4^2)(\bar{\lambda}_3 - \bar{\lambda}_4)} \right) \sin^2 \phi \\ & + \left( \frac{1}{(1 + \bar{\lambda}_2^2)(\bar{\lambda}_3 - \bar{\lambda}_2)} + \frac{1}{(1 + \bar{\lambda}_5^2)(\bar{\lambda}_3 - \bar{\lambda}_5)} \right) \cos^2 \phi = 0. \end{aligned}$$

Note that  $\bar{\lambda}_1 \bar{\lambda}_4 = -1$  and  $\bar{\lambda}_2 \bar{\lambda}_5 = -1$ . Hence (36) is equivalent to

$$(37) \quad \frac{\bar{\lambda}_3 - \bar{\lambda}_1 - \bar{\lambda}_4}{(\bar{\lambda}_3 - \bar{\lambda}_1)(\bar{\lambda}_3 - \bar{\lambda}_4)} \sin^2 \phi + \frac{\bar{\lambda}_3 - \bar{\lambda}_2 - \bar{\lambda}_5}{(\bar{\lambda}_3 - \bar{\lambda}_2)(\bar{\lambda}_3 - \bar{\lambda}_5)} \cos^2 \phi = 0,$$

which has solutions if

$$(38) \quad (\bar{\lambda}_3 - \bar{\lambda}_1)(\bar{\lambda}_3 - \bar{\lambda}_4)(\bar{\lambda}_3 - \bar{\lambda}_1 - \bar{\lambda}_4)(\bar{\lambda}_3 - \bar{\lambda}_2)(\bar{\lambda}_3 - \bar{\lambda}_5)(\bar{\lambda}_3 - \bar{\lambda}_2 - \bar{\lambda}_5) < 0.$$

Note that  $\bar{\lambda}_3 < \bar{\lambda}_2 < \bar{\lambda}_1$ ,  $\bar{\lambda}_3 > \bar{\lambda}_4 > \bar{\lambda}_5$ . By Lemma 2.5, we know that (38) holds if  $\frac{1}{\sqrt[6]{3}} < \bar{\lambda}_2 < \sqrt[6]{3}$ .

Let  $\phi_0$  be a solution of (37). It follows from (33) that there is an  $x_0$  in  $M$  such that

$$\begin{aligned} a_1^2(z) &= \frac{\sin^2 \phi_0}{1 + \bar{\lambda}_1^2}, & a_2^2(z) &= \frac{\cos^2 \phi_0}{1 + \bar{\lambda}_2^2}, & a_3(z) &= 0, \\ a_4^2(z) &= \frac{\sin^2 \phi_0}{1 + \bar{\lambda}_4^2}, & a_5^2(z) &= \frac{\cos^2 \phi_0}{1 + \bar{\lambda}_5^2}, & a_6(z) &= 0 \end{aligned}$$

for any  $z \in \pi^{-1}(x_0)$ . Then  $\bar{\lambda}_3$  is a principal curvature with multiplicity 2 of  $M$  at  $x_0$ , and  $g(x_0) = 4$ . Similarly, we can prove that there is an  $x_0$  in  $M$  such that  $\bar{\lambda}_4$  is

a principal curvature with multiplicity 2 if  $\frac{\sqrt{3}}{3} < \bar{\lambda}_2 < \sqrt{3}$ . Hence  $M$  has 5 principal curvatures at general points and 3 principal curvatures at special points.  $\square$

Combining Theorems 1.2, 1.3 and Propositions 2.1, 2.2, 2.3 and 2.4, we have our main result:

**Theorem.** *Let  $M$  be a complete isoparametric hypersurface in  $\mathbb{C}P^n$ . Then  $M$  is homogeneous if and only if either  $g$  or  $l$  is constant.*

# REFERENCES

1. U. Abresch, Isoparametric hypersurfaces with four or six distinct principal curvatures, *Math. Ann.* 264(1983), 283-302. MR **85g**:53052b
2. E. Cartan, Familles des surfaces isoparametriques dans les espace a courbure constante, *Ann. Mat. Pura Appl.* (4) 17(1938) 177-191.
3. J. Dorfmeister and E. Neher, Isoparametric hypersurfaces, case  $g = 6, m = 1$ , *Comm. in Alg.* 13(1985), 2299-2368. MR **87d**:53096
4. M. Kimura, Real hypersurfaces and complex submanifolds in complex projective space, *Tran. Amer. Math. Soc.* 296 (1986) 137-149. MR **87k**:53133
5. H.F. Münzner, Isoparametrische Hyperflächen in Sphären. I. *Math. Ann.* 251 (1980) 57-71. MR **82a**:53058
6. H.F. Münzner, Isoparametrische Hyperflächen in Sphären. II. *Math. Ann.* 256 (1981) 215-232. MR **82m**:53053
7. B. O'Neill, Submersions and geodesics, *Duke Math. J.* 34 (1967) 363-373. MR **35**:7265
8. K.S. Park, Isoparametric families on projective spaces, *Math. Ann.* 284 (1989) 503-513. MR **90f**:53095
9. C.K. Peng and Z.X. Hou, A remark on the isoparametric polynomials of degree 6, *Lect. Notes in Math.* 1369 (1989), 222-224. MR **90g**:53068
10. R. Takagi, A class of hypersurfaces with constant principal curvatures in a sphere, *J. Diff. Geo.* 11 (1976), 225-233. MR **54**:13798
11. R. Takagi, On homogeneous real hypersurfaces in a complex projective space, *Osaka J. Math.* 10 (1973) 495-506. MR **49**:1433
12. R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures, *J. Math. Soc. Japan* 27(1975) 43-53. MR **50**:8380
13. R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures, *J. Math. Soc. Japan* 27(1975) 507-516. MR **53**:3955
14. C.L. Terng and G. Thorbergsson, Submanifold geometry in symmetric spaces, *J. Diff. Geom.* 42 (1995), 665-718. MR **97k**:53054
15. Q. Wang, Isoparametric hypersurfaces in complex projective spaces, *Proc. of the 1980 Beijing Symp. on Differential Geometry and Differential Equations*, Vol. 1, 2, 3 (Beijing 1980) 1509-1523, Science Press, Beijing 1982. MR **85c**:53094
16. L. Xiao, Isoparametric submanifolds in complex projective spaces (in Chinese), *Acta Math. Sinica* 38(1995) no.6 845-856. MR **97e**:53110
17. L. Xiao, Polynomial representations of homogeneous isoparametric submanifolds (in Chinese), *Adv. in Math. (China)* 24 (1995) 145-154. MR **96d**:53052

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA (BEIJING), P.O. BOX 3908, BEIJING 100039, P.R.CHINA  
*E-mail address:* lxiao@tonghua.com.cn